Properties of integrals and u-substitution: 2 problems from Reddit

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The rules are simple. Sometimes, the rules show up in very interesting ways.

Question.

Compute:

$$\int_1^2 x^9 \; f(2x^5 - 5) \; dx \tag{1}$$

and

$$\int_2^7 f\left(\frac{5}{x}\right) \; dx \tag{2}$$

given

$$\int_{\frac{5}{2}}^{\frac{5}{7}} \frac{f(x)}{x^2} \; dx = 13 \qquad \int_2^7 \frac{f(x)}{x^2} \; dx = 28$$

$$\int_{\frac{5}{2}}^{\frac{5}{7}} \frac{f(x)}{x} \; dx = 36 \qquad \int_{-3}^{59} x f(x) \; dx = 46$$

$$\int_{-3}^{59} f(x) \; dx = 9$$

Solution.

There appears to be a lot happening here! Not the least of which is that we do not know $f(x)$, and there appears to be no easy way to find it.

At this point, we can examine what we're given and see if there are some signals about where we might start. In the integrals we're asked to compute, we do have composite functions, which should suggest that we might be looking at undoing the chain rule. Undoing the chain rule on an integrand is integration by substitution, and we commonly use a u-substitution to get that done. In fact, this is a great place to start.

Let's start by looking at (2). So, our "inside" function is $\frac{5}{x}$, so let's proceed as usual with a u-substitution:

$$u = \frac{5}{x}$$

$$\frac{du}{dx} = -5x^{-2} \qquad \text{(differentiate)}$$

$$dx = -\frac{1}{5}x^2\ du \qquad \text{(rearrange)}$$

Next, we must transform our limits of integration by expressing a and b in terms of u:

$$a = 2 \qquad \text{(in terms of } x\text{)}$$

$$a = \frac{5}{2} \qquad \text{(in terms of } u \text{ by substitution)}$$

$$b = 7 \qquad \text{(in terms of } x\text{)}$$

$$b = \frac{5}{7} \qquad \text{(in terms of } u \text{ by substitution)}$$

Taking our integral now, in terms of u, we have:

$$\int_2^7 f\left(\frac{5}{x}\right)\ dx = \int_{\frac{5}{2}}^{\frac{5}{7}} \left(-\frac{1}{5}x^2\right) f(u)\ du \qquad \text{(substitute)}$$

$$= -\frac{1}{5}\int_{\frac{5}{2}}^{\frac{5}{7}} x^2 f(u)\ du \qquad \text{(factor constant)}$$

Since we let $u = \frac{5}{x}$, then we can rearrange to give us $x = \frac{5}{u}$. Substituting, we have:

$$-\frac{1}{5}\int_{\frac{5}{2}}^{\frac{5}{7}} x^2 f(u)\ du = -\frac{1}{5}\int_{\frac{5}{2}}^{\frac{5}{7}} \left(\frac{5}{u}\right)^2 f(u)\ du$$

$$= -\frac{25}{5}\int_{\frac{5}{2}}^{\frac{5}{7}} \frac{f(u)}{u^2}\ du$$

$$= -5\int_{\frac{5}{2}}^{\frac{5}{7}} \frac{f(u)}{u^2}\ du$$

Sure enough, this is one of our given integrals. The variable does not matter at all, and our integral in u is the very same as one of the ones given in x, limits and all. We can finally say:

$$-5\int_{\frac{5}{2}}^{\frac{5}{7}} \frac{f(u)}{u^2}\ du = -5\cdot 13$$

$$= -65$$

Done.

Now, let's have a look at (1). Same idea, slightly more complicated, but not by much. We do have an "inside" function again, so let's proceed similarly. Let:

$$u = 2x^5 - 5 \qquad \text{(substitute)}$$

$$\frac{du}{dx} = 10x^4 \qquad \text{(differentiate)}$$

$$dx = \frac{1}{10x^4}\ du \qquad \text{(rearrange)}$$

Transforming our limits of integration:

$$\begin{aligned} a &= 1 && \text{(in terms of } x\text{)} \\ a &= 2(1)^5 - 5 && \text{(in terms of } u\text{, by substitution)} \\ &= -3 \end{aligned}$$

$$\begin{aligned} b &= 2 && \text{(in terms of } x\text{)} \\ b &= 2(2)^5 - 5 && \text{(in terms of } u\text{, by substitution)} \\ &= 59 \end{aligned}$$

So, we have:

$$\int_1^2 x^9\ f(2x^5 - 5)\ dx = \int_{-3}^{59} \frac{x^9}{10x^4}\ f(u)\ du \qquad \text{(substitute)}$$

$$= \frac{1}{10}\int_{-3}^{59} x^5 f(u)\ du \qquad \text{(simplify)}$$

To deal with the x^5, we use that $u = 2x^5 - 5$, so that we can say that $x^5 = \frac{1}{2}(u+5)$:[1]

$$\frac{1}{10}\int_{-3}^{59} x^5 f(u)\ du = \frac{1}{10}\int_{-3}^{59} \frac{1}{2}(u+5)f(u)\ du \qquad \text{(substitute)}$$

$$= \frac{1}{20}\int_{-3}^{59} (u+5)f(u)\ du \qquad \text{(factor out constants \& simplify)}$$

[1]Note that we don't have to solve for x, rather, solving for x^5 saves a calculation or two!

$$= \frac{1}{20}\left[\int_{-3}^{59} uf(u)\ du + 5\int_{-3}^{59} f(u)\ du\right] \qquad \text{(distribute integral, using linearity)}$$

And, again, we see that, in the square brackets, we have two of the integrals for which we are given values in the problem statement. We can proceed:

$$\frac{1}{20}\left[\int_{-3}^{59} uf(u)\ du + 5\int_{-3}^{59} f(u)\ du\right] = \frac{1}{20}[46 + 5 \cdot 9] \qquad \text{(use given information \& substitute)}$$

$$= \frac{91}{20} \qquad \text{(simplify)}$$

Done.

Note that for these two questions, we did not need to know $f(x)$, the integrand, nor $F(x)$, the antiderivative to solve! Super interesting!

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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