An interesting Mean Value Theorem question about a piecewise function

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This is a really interesting question from the front lines of AP Calculus AB.

Question.

Find the values of a, b, and c that make

$$f(x) = \begin{cases} 5, & x = 0 \\ ax + b, & 0 < x \leq 1 \\ x^2 + 3x + c, & 1 < x \leq 5 \end{cases}$$

satisfy the requirements of the Mean Value Theorem.

Solution.

This *looks* sort of innocuous, doesn't it? It has its share of subtleties and is super interesting!

Let's, of course, start with the Mean Value Theorem:

Theorem 1. *If f is a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that the tangent at c is parallel to the secant line through the endpoints $(a, f(a))$ and $(b, f(b))$.*[1]

The *conditions* of the Mean Value Theorem that need to be met are **continuity** and **differentiability** over the open interval we're considering. In the definition of $f(x)$, we see that our open interval is $x \epsilon (0, 5)$.

We can think of ***continuity***, in this case, as "can I draw the graph without lifting my pencil?" Yes, we can. Have a look at the domains for the different pieces and note that there are no holes. That's probably a specific-enough definition of continuity for this problem, but let's consider it algebraically. In the definition of $f(x)$, we have a point, a line, and a parabola over their respective domains. For continuity, these must be true:

$$f(0) = 5 \qquad \text{(trivial, this is the point } (0,5)\text{)} \tag{1}$$

$$ax + b = f(0) \qquad \text{(needed for continuity between first two pieces)} \tag{2}$$

$$x^2 + 3x + c = f(1) \qquad \text{(needed for continuity between last two pieces)} \tag{3}$$

Convince yourself these have to be true, and, specifically, think of drawing the graph with holes on the boundaries between pieces and see that the holes become filled in as we graph.

The system of equations above degenerates a bit, since we can deduce that $b = 5$ has to be true for the $ax + b$ portion of $f(x)$, that is, the line $ax + b$ has to have a y-intercept of 5. We hit a wall at this point, on continuity alone, because, for $x = 1$:

[1] https://en.wikipedia.org/wiki/Mean_value_theorem

$$
\begin{aligned}
ax + 5 &= x^2 + 3x + c && \text{(linear and quadratic pieces have to have the same } y = f(1) \text{ value)} \\
a \cdot 1 + 5 &= 1^2 + 3 \cdot 1 + c && \text{(substitute 1 for } x\text{)} \\
a + 5 &= 4 + c && \text{(simplify)} \\
c - a &= 1 && (4)
\end{aligned}
$$

Too many variables, not enough equations.

Now, let's consider ***differentiability***. Just like we need common points where the pieces intersect, we also need common tangent lines at those values of x. Let's start by taking the first derivative:

$$f'(x) = \begin{cases} 0, & x = 0 \\ a, & 0 < x \leq 1 \\ 2x + 3, & 1 < x \leq 5 \end{cases}$$

The only place we can have a tangent line to examine is at $x = 1$, since the Mean Value Theorem deals with open intervals.[2] Because the middle piece of $f(x)$ is a line, the value of the derivative of $f(x)$ at $x = 1$ has to be the same as the slope of line, namely $f'(x) = a$ (everywhere!) for the length of the linear segment. In order for the $f(x)$ to be *differentiable* at $x = 1$ the tangent lines have to be the same, and, more specifically, the *slopes* of the tangents of the linear and quadratic pieces of $f(x)$ have to be equal at $x = 1$. So:

$$
\begin{aligned}
a &= 2x + 3 && \text{(this has to be true at } x = 1\text{)} \\
a &= 2(1) + 3 && \text{(substitute)} \\
a &= 5 && (5)
\end{aligned}
$$

Recalling (4), we have $c - a = 1$, making $c = 6$. Our function is therefore:

$$f(x) = \begin{cases} 5, & x = 0 \\ 5x + 5, & 0 < x \leq 1 \\ x^2 + 3x + 6, & 1 < x \leq 5 \end{cases}$$

I made a Desmos[3] workbook, `https://www.desmos.com/calculator/on0nqldwde`, that shows our function, with the proper constants, and what it looks like when we graph it. I used dotted and dashed lines to show the various pieces. When you bring up this workbook, one curve is turned off: the tangent line at $x = 1$. Turn it on and zoom in to see what's happening, and also note the point-slope formula for the equation of the tangent line at $(x_1, f(x_1))$:

$$y - f(x_1) = f'(x_1)(x - x_1)$$

This is a really good question about a piecewise function. I say so mostly because it is somewhat difficult to "eyeball" and guess at the solution; we actually have to do the algebra here, with not much more to guide us. The picture can verify our algebra, which I believe Desmos does for us in this case.

Done.

[2] considering open intervals means that we don't have to worry about the endpoints. Here, the endpoints of the interval are $x = 0$ and $x = 5$.

[3] `http://www.desmos.com/calculator`

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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Thank you so much.