Same integral, two ways to solve

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There are many ways to deal with integration problems. I got this one from Reddit. The problem demanded that it be solved as a trigonometric substitution, but it's trivial if you use a u-substitution! Let's have a look.

Question.

Solve the following:

$$\int x\sqrt{36 - x^2}dx$$

.

Solution, using u-substitution.

On inspection, we can see that this integral is a great candidate for a u-substitution, as we have both a function $(36 - x^2)$ and its derivative (x) present in the integrand. So, let:

$$\begin{aligned} u &= 36 - x^2 \\ du &= -2xdx \end{aligned}$$

$$\int \sqrt{u} \cdot x\frac{du}{-2x} = -\frac{1}{2}\int \sqrt{u} \cdot du \qquad \text{(substitute \& simplify)}$$

$$= -\frac{1}{3}u^{\frac{3}{2}} \qquad \text{(integrate)}$$

So:

$$\int x\sqrt{36 - x^2}dx = -\frac{1}{3}(36 - x^2)^{\frac{3}{2}} + C$$

Done.

Solution, using trig substitution.

Also on inspection, we might notice that the radical is a classic form that usually calls for a trig substitution. As shown above, the x outside of the radical doesn't make it *necessary*, but a trig substitution is *possible*. Let:

$$x = 6sin\theta$$
$$dx = 6cos\theta d\theta$$
$$x^2 = 36sin^2\theta$$

So:

$$\int 6sin\theta\sqrt{36 - sin^2}6cos\theta d\theta = 6^3 \int sin\theta cos^2\theta d\theta \qquad \text{(substitute \& simplify)}$$

Cleaned up kind of nicely. With this integral in terms of theta, we see that a u-substitution is in order. Let:

$$u = cos\theta$$
$$du = -sin\theta d\theta$$

So:

$$6^3 \int sin\theta u^2 \frac{du}{-sin\theta} = -6^3 \int u^2 du \qquad \text{(substitute \& simplify)}$$

$$= -\frac{6^3}{3}u^3 \qquad \text{(integrate and simplify)}$$

Easy enough, but now we have to undo *two* substitutions:

$$-\frac{6^3}{3}u^3 = -\frac{6^3}{3}cos^3\theta \qquad \text{(undo the u-sub)}$$

$$= -\frac{6^3}{3}cos^3\left[sin^{-1}\left(\frac{x}{6}\right)\right] + C \qquad \text{(undo the trig sub)}$$

So:

$$\int x\sqrt{36 - x^2}dx = -\frac{6^3}{3}cos^3\left[sin^{-1}\left(\frac{x}{6}\right)\right] + C$$

Done.

Examining both answers.

OK, so both processes yielded answers:

$$\int x\sqrt{36-x^2}dx = -\frac{1}{3}(36-x^2)^{\frac{3}{2}} + C \qquad \text{(via u-substitution)}$$

and:

$$\int x\sqrt{36-x^2}dx = -\frac{6^3}{3}cos^3\left[sin^{-1}\left(\frac{x}{6}\right)\right] + C \qquad \text{(via trig substitution)}$$

But, are these the same? We could differentiate both and see if we get the original integrand, but looking at the graphs of each would work just as well. I shared the graphs of both: `https://www.desmos.com/calculator/gnnlvwrmxt`. You can click on the blue and red icons on the left to turn the graphs on and off. They're the same!

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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