Integration! Odd functions! Even functions! Calculus Thinking! A weird problem!

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March 4, 2021

I tried so many things on and off for a few weeks on this one! Then I got my head straight and solved it in what felt like a matter of minutes. I probably wrote too much for this one, but I wanted to give my readers a sense of how, oftentimes, good problem solving consists of many twists and turns. Sometimes, unexpected "tries" at something can reveal "hidden" details.

Question.

Integrate the following:

$$\int_{-2}^{2} \frac{1+x^2}{1+2^x}\, dx$$

Solution.

1 Struggle ... then, breakthrough

Experienced Calculus practitioners notice a tough issue straight away: that 2^x term in the denominator. Can be a bad sign sometimes.

I think it's best to start with a definition that I use with my students all the time:

Definition 1. *Calculus Thinking*

> ***A problem-solving state of mind*** *that we assume when a problem looks frightening at first blush: We try to rephrase all or part of the problem in ways that are easier to recognize, that are more tractable, easier to solve. This could - and usually does - involve terrifying algebra (or trig, or ...), but! It will be worth it. The intuition and elegance that we show off in this process is the essence of* ***Calculus Thinking****.*

To be specific, Calculus Thinking is the main skill we employ when dealing with, say, indeterminate forms of limits *without* L'Hôpital's Rule, dealing with most nontrivial Integration problems, doing applications problems of all types, etc. Its value cannot be understated, in my opinion. I'd go as far as to say that when we learn Calculus, we must master Calculus Thinking: being in a place where we are able to take advantage of every math trick we've ever learned, refreshed on every math trick we've *forgotten*, and being able to employ *any* of them to throw some shapes around, in the interest of *just seeing what happens*.

So what does this have to do with this integral that we're considering? Having a closer look, we notice that the interval over which we're integrating is symmetrical about the y-axis and includes the origin: $x \epsilon [-2, 2]$. For me, this changed my whole way of thinking around this problem once I asked myself if the odd or even nature of this $f(x) = \frac{1+x^2}{1+2^x}$ could help it be more tractable.

2 Odd and even functions, or, remember your Algebra II

2.1 Definitions

Definition 2. *Even function*

> *For any function f(x), if $f(-x) = f(x)$, then the function is called **even**.*

Corollary 1. *Functions that are even are sometimes described as having **y-axis symmetry**.*

Common examples of even functions include polynomials of degree n (for even integers, n), cosine, absolute value.

Definition 3. *Odd function*

> *For any function f(x), if $f(-x) = -f(x)$, then the function is called **odd**.*

Corollary 2. *Functions that are odd are sometimes described as having **origin symmetry**.*

Common examples of odd functions include polynomials of degree n (for odd integers, n), sine, hyperbolic sine.

2.2 Speaking of Algebra II, rational functions suck

(Well, they're unpleasant, anyway.) Calculus makes many things easier, sometimes generalizes topics, and makes solutions to many things proceed in interesting ways. Except for rational functions. They still kinda suck. Too many details to juggle. But, it's part of the practice, as rational functions are still the domains of many relationships in nature.

2.3 Some Calculus implications of oddness and evenness

If we think strictly of areas (and therefore integrals), there are interesting characteristics of both odd and even functions:

Theorem 1. *For any symmetrical interval I: $x \epsilon [-a, a]$ over which f is continuous, the integral of an odd function is zero. That is:*

$$\int_{-a}^{a} f(x)\ dx = 0$$

This makes sense, if we think about it. For an odd function $f(x)$, there are equal areas bounded by the curve and the x-axis: one above, one below. These add up to give us zero. Try it with $f(x) = sinx$ and any other odd functions you know.

Theorem 2. *For any symmetrical interval I: $x \epsilon [-a, a]$ over which f is continuous, the integral of an even function is:*

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

This makes sense, too. For an even function $f(x)$, there are equal areas on either side of the y-axis: both above or both below.

3 Is our integrand an odd function or an even function?

3.1 Test for odd

For our function to be odd, $f(-x) = -f(x)$ has to hold:

$$f(-x) \stackrel{?}{=} -f(x)$$

$$\frac{1+(-x)^2}{1+2^{-x}} \stackrel{?}{=} -\frac{1+x^2}{1+2^x}$$

$$\frac{1+x^2}{1+\frac{1}{2^x}} \stackrel{?}{=}$$

$$\frac{1+x^2}{\frac{2^x+1}{2^x}} \stackrel{?}{=}$$

$$\frac{2^x(1+x^2)}{2^x+1} \stackrel{?}{=}$$

$$2^x \cdot \frac{1+x^2}{2^x+1} \stackrel{?}{=} \qquad \text{(rewrite)}$$

With a final rewrite, we see:

$$2^x \cdot \frac{1+x^2}{1+2^x} \neq -\frac{1+x^2}{1+2^x} \tag{1}$$

So, our function, $f(x)$, does not appear to be odd. Fair enough.

3.2 Test for even

For our function to be even, $f(-x) = f(x)$ has to hold:

$$f(-x) \stackrel{?}{=} f(x)$$

$$\frac{1+(-x)^2}{1+2^{-x}} \stackrel{?}{=} \frac{1+x^2}{1+2^x}$$

We can quickly borrow our results from (1), above. Left-hand side is the same, and right-hand side is only slightly different (no minus sign for the whole fraction):

$$2^x \cdot \frac{1+x^2}{1+2^x} \stackrel{?}{=} \frac{1+x^2}{1+2^x}$$

This is *certainly* not true either, so:

$$2^x \cdot \frac{1+x^2}{1+2^x} \neq \frac{1+x^2}{1+2^x} \tag{2}$$

3.3 So. Not even. Not odd. Why did I even bother?

Simply put: *because I* ***could***. And this is where Calculus Thinking comes in. There is an interesting result in (1):

$$f(-x) = 2^x \cdot \frac{1 + x^2}{1 + 2^x}$$

Which simply means:

$$f(-x) = 2^x \cdot f(x) \tag{3}$$

3.4 Back to even and odd functions for a sec

From `https://en.wikipedia.org/wiki/Even_and_odd_functions`:

Theorem 3. *Every function may be uniquely decomposed as the sum of an even and an odd function, which are called, respectively, the* ***even part*** *and the* ***odd part*** *of the function.*

Specifically:

$$f(x) = f_e(x) \ + f_o(x) \tag{4}$$

where $f_e(x)$ is the even part of $f(x)$ and $f_o(x)$ is the odd part of $f(x)$. This has some interesting implications.

When $f(x)$ is even, we know $f(x) = f(-x)$, and we can also say that $f_o(x)$ is necessarily zero, i.e., $f(x)$ has no odd part; it's solely even. So:

$$\begin{aligned} \text{if} : f(x) &= f_e(x) \\ \text{then} : f(-x) &= f_e(x) && \text{(substitution)} \end{aligned}$$

and:

$$2f_e(x) = f(x) + f(-x) \qquad \text{(add equations)}$$

So:

$$f_e(x) = \frac{1}{2}\big[f(x) + f(-x)\big] \tag{5}$$

A similar process can be done for $f_o(x)$ and results in:

$$f_o(x) = \frac{1}{2}\big[f(x) - f(-x)\big] \tag{6}$$

Please prove for yourself that these satisfy (4).

To get really specific about what these results mean, consider:

$$\int_{-a}^{a} f(x)\ dx = \int_{-a}^{a} f_e(x) + f_o(x)\ dx$$

Applying what we've learned so far, we can say:

$$\int_{-a}^{a} f(x)\ dx = \int_{-a}^{a} f_e(x) + f_o(x)\ dx$$

$$= \int_{-a}^{a} f_e(x) + f_o(x)\ dx$$

$$= \int_{-a}^{a} f_e(x)\ dx + \int_{-a}^{a} f_o(x)\ dx$$

$$= 2\int_{0}^{a} f_e(x)\ dx + 0$$

So:

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f_e(x)\ dx \tag{7}$$

Quite an interesting result. So, we can say:

Corollary 3. *to Theorem 3*
To integrate any function, f(x) over a symmetrical interval $x \epsilon [-a, a]$ where f is continuous, all we have to do is integrate the ***even*** *component of f(x) – $f_e(x)$ – over the interval $x \epsilon [0, a]$ and multiply by 2.*

So, we have the following:

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f_e(x)\ dx$$

$$= 2\int_{0}^{a} \frac{1}{2}\big[f(x) + f(-x)\big]\ dx \qquad \text{(substituting } f_e(x) \text{ from above)}$$

$$= \int_{0}^{a} f(x)\ dx + \int_{0}^{a} f(-x)\ dx \qquad \text{(clean up and expand)}$$

So, for any function $f(x)$ integrated over $x \epsilon [-a, a]$:

$$\int_{-a}^{a} f(x)\ dx = \int_{0}^{a} f(x)\ dx + \int_{0}^{a} f(-x)\ dx \tag{8}$$

It's actually a substitution of sorts!

3.4.1 A Desmos workbook for even/odd decomposition of functions

To help with future problems, I made a Desmos workbook for decomposing functions into their even and odd components: `https://www.desmos.com/calculator/vsi3n8ej7l`. All you have to do is replace $f(x)$ with the function you wish to decompose. $f_e(x)$ and $f_o(x)$ are calculated for you! Note that if your $f(x)$ results in $f_e(x) = 0$, your function is strictly odd, that is, there is no even component. Similarly, for a function that is strictly even, $f_o(x)$ will be zero.

4 Now we have everything needed to solve our integral

We want to integrate:

$$\int_{-2}^{2} \frac{1+x^2}{1+2^x}\ dx$$

using our result, (8). We need $f(-x)$ to do so, so recall our result from (3), above:

$$f(x) = \frac{1+x^2}{1+2^x}$$

$$f(-x) = 2^x \cdot f(x)$$

Thinking ahead to substituting, let's choose to use a slightly different version of (8), namely:

$$\int_{-a}^{a} f(x)\ dx = \int_{0}^{a} f(x) + f(-x)\ dx \qquad \text{(just a rewrite of (8))}$$

Substituting, we have:

$$\int_{-2}^{2} \frac{1+x^2}{1+2^x}\ dx = \int_{0}^{a} \left[\frac{1+x^2}{1+2^x}\right] + 2^x \cdot \left[\frac{1+x^2}{1+2^x}\right]\ dx$$

$$= \int_{0}^{2} \left[\frac{1+x^2}{1+2^x}\right] \cdot (1+2^x)\ dx \qquad \text{(factor the integrand)}$$

$$= \int_{0}^{2} 1+x^2\ dx \qquad \text{(cancel and simplify! OMG!)}$$

$$= x + \frac{1}{3}x^3\Big]_0^2$$

$$= 2 + \frac{8}{3}$$

$$= \frac{14}{3}$$

Done. Did NOT see that coming.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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