Advanced factoring trick for a tough limit problem

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Sometimes, limit problems can be difficult. This is one such limit where an advanced factoring trick unlocks the solution.

Question.

Calculate:

$$L = \lim_{x \to -\infty} \frac{|2 - x|}{x + \sqrt{\frac{x^2}{4} + x}}$$

Solution.

There is so much going on in this problem! A rational function, an absolute value, and x approaching $-\infty$. In fact, these three details are the ones that will drive the solution of this problem; they make up the signals that help determine how to proceed.

Limits at infinity can be difficult. So can rational functions. We might be tempted to use the conjugate trick to massage this fraction, but it likely will not help with the absolute value term in the numerator. That being said, there are some facts and definitions that we will use while solving this problem.

One broad limit-related fact we need to keep in mind here is:

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0$$

In addition, we will make use of the definition of absolute value:

$$|x| = \begin{cases} -1 \cdot x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

We are also going to use a corollary to the definition of $|x|$, which is:

$$\frac{x}{|x|} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

Now that we've got these things in mind, let's begin.

1 There is a lot going on in the denominator

Straight away, this is where and how we are going to blow this problem wide open. There is a signal in that x^2 term, and that is to *factor* the expression under the radical, so let's start there.

$$\begin{aligned} L &= \lim_{x\to-\infty} \frac{|2-x|}{x+\sqrt{\frac{x^2}{4}+x}} \\ &= \lim_{x\to-\infty} \frac{|2-x|}{x+\sqrt{x^2\left(\frac{1}{4}+\frac{1}{x}\right)}} && \text{(factor under the radical)} \\ &= \lim_{x\to-\infty} \frac{|2-x|}{x+\sqrt{x^2}\cdot\sqrt{\left(\frac{1}{4}+\frac{1}{x}\right)}} && \text{(clarify the factoring a bit)} \end{aligned}$$

This looks so messy on paper, but we are getting there. One thing to note at this point: since x is approaching $-\infty$, we have to show that simplifying $\sqrt{x^2}$ (taking the positive square root) in such a way that we retain the "negativeness" of x. When we square something, recall that we "lose the sign". Here, x is definitely negative, again, as we head to $-\infty$. We can use $|x|$ to retain that x is negative, whilst producing the positive square root correctly. This leaves us:

$$\begin{aligned} L = &\lim_{x\to-\infty} \frac{|2-x|}{x+\sqrt{x^2}\cdot\sqrt{\left(\frac{1}{4}+\frac{1}{x}\right)}} \\ = &\lim_{x\to-\infty} \frac{|2-x|}{x+|x|\cdot\sqrt{\left(\frac{1}{4}+\frac{1}{x}\right)}} && \text{(simplify square root by introducing absolute value)} \end{aligned}$$

At this point, we will use our cool factoring trick. We can use the definition of absolute value and the fact that x is negative, to properly factor out the $|x|$ term in the following way:

$$\begin{aligned} L = &\lim_{x\to-\infty} \frac{|2-x|}{x+|x|\cdot\sqrt{\left(\frac{1}{4}+\frac{1}{x}\right)}} \\ = &\lim_{x\to-\infty} \frac{|2-x|}{|x|\left(-1+\sqrt{\left(\frac{1}{4}+\frac{1}{x}\right)}\right)} && \text{(factor out the } |x| \text{ term)} \end{aligned}$$

1.1 Looking at the trick in more detail

Take a moment to convince yourself that this is factored correctly. To start, we can think of factoring as dividing each term by the term that is factored out. Let's look at the denominator (I'll call it D) and show the factoring in a way that clarifies what's happening. We started with:

$$D = x + |x| \cdot \sqrt{\left(\frac{1}{4} + \frac{1}{x}\right)}$$

$$= |x|\left(\frac{x}{|x|} + \frac{|x|}{|x|}\sqrt{\left(\frac{1}{4} + \frac{1}{x}\right)}\right) \qquad \text{(factor out } |x| \text{ by dividing each term by it)}$$

$$= |x|\left(-1 + 1 \cdot \sqrt{\left(\frac{1}{4} + \frac{1}{x}\right)}\right) \qquad \text{(simplify using the facts about absolute value stated above)}$$

$$D = |x|\left(-1 + \sqrt{\left(\frac{1}{4} + \frac{1}{x}\right)}\right) \qquad \text{(simplify)}$$

2 Using limit properties to carry this to the finish line

We can start by rearranging the terms in our limit and go from there:

$$L = \lim_{x \to -\infty} \frac{|2 - x|}{|x|\left(-1 + \sqrt{\left(\frac{1}{4} + \frac{1}{x}\right)}\right)}$$

$$= \lim_{x \to -\infty} \frac{|2 - x|}{|x|} \cdot \frac{1}{-1 + \sqrt{\frac{1}{4} + \frac{1}{x}}} \qquad \text{(rearrange to bring the absolute value terms together)}$$

$$= \lim_{x \to -\infty} \left|\frac{2 - x}{x}\right| \cdot \frac{1}{-1 + \sqrt{\frac{1}{4} + \frac{1}{x}}} \qquad \text{(use absolute value quotient property)}$$

$$= \lim_{x \to -\infty} \left|\frac{2}{x} - 1\right| \cdot \frac{1}{-1 + \sqrt{\frac{1}{4} + \frac{1}{x}}} \qquad \text{(divide)}$$

Now we are ready to evaluate our limit:

$$L = \lim_{x \to -\infty} \left|\frac{2}{x} - 1\right| \cdot \frac{1}{-1 + \sqrt{\frac{1}{4} + \frac{1}{x}}}$$

$$= |-1| \cdot \frac{1}{-1 + \sqrt{\frac{1}{4}}} \qquad \text{(evaluate the limit)}$$

$$= \frac{1}{-1 + \frac{1}{2}} \qquad \text{(simplify...)}$$

$$= \frac{1}{-\frac{1}{2}}$$

$$L = -2$$

Done.

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