An interesting integration question: cutting an area in half with a line

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April 23, 2021

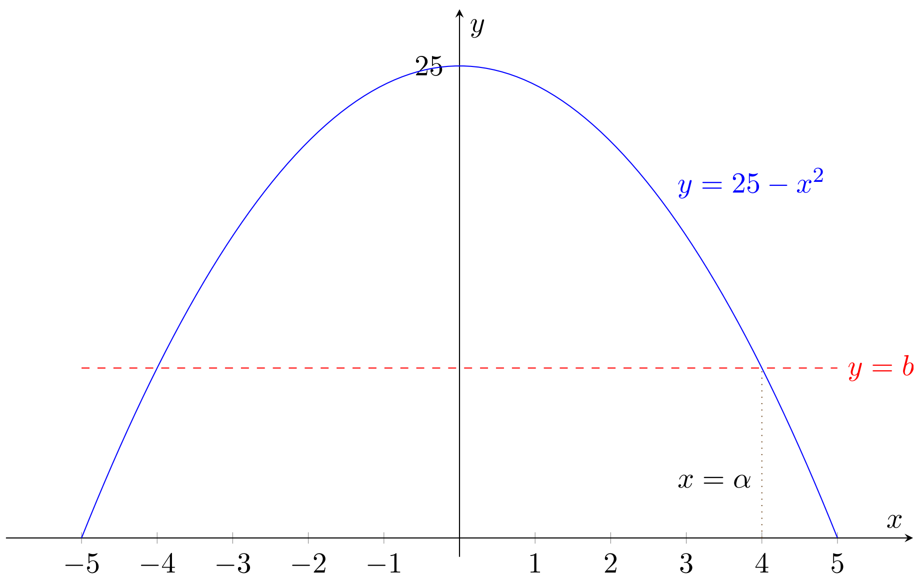
Another cool integration question from the front lines of Calc I / AP Calc.

Question.

Find a horizontal line, $y = b$, that divides the area under the curve of $f(x) = 25 - x^2$ in half.

Solution.

A great place to start is to have a look at the graph $f(x)$:



To be honest, this one was a bit weird for me to visualize, at first. I had some sort of sketch like the one above, but I felt like I was always missing one thing. The essence of the problem is this:

> How can we slide $x = \alpha$, which intersects both $y = b$ and $y = f(x)$ at the same point, left and right, such that $y = b$ cuts the area in half?

Quite a mouthful, but it contains the relevant information. All we have to do is translate this into Calculus symbols and get solving!

We have, $f(x) = 25 - x^2$. With a little **Calculus Thinking**,[1] we can employ a trick that takes advantage of the symmetry of the graph of $f(x)$: all we really need to do is consider the interval $x \, \epsilon \, [0, 5]$, in order to place $y = b$

[1] `https://mymathteacheristerrible.com/blog/2019/5/15/calculus-thinking-a-definition-i-use-over-and-over-with-my-students`

correctly. Please verify to yourself that if we take *half of a half* of the area we're considering, it will yield the same solution, which places $y = b$ properly. If $y = b$ is placed properly for the right half of the curve, it will still be placed properly when looking at the area in question. When we write out our integrals, we will see this trick pay off!

So the area under $f(x)$ on the interval $[0, 5]$ is:

$$A_{f(x)} = \int_0^5 f(x)\ dx$$

and, as far as $y = b$ is concerned, we have to consider two areas: the area under $y = b$ to the left of $x = \alpha$ (the interval $[0, \alpha]$) and then the little piece to the right of $x = \alpha$, which is actually the area under $f(x)$ on the interval $[\alpha, 5]$. *This* is the slightly weird part of this problem, IMO. We can write this out way more clearly using integrals:

$$A_{y=b} = \int_0^\alpha b\ dx + \int_\alpha^5 f(x)\ dx$$

But? Do we really need that first integral? Take a shortcut: the area under $y = b$ on $[0, \alpha]$ is clearly a rectangle, so its area is $\alpha \cdot b$. This gives us

$$A_{y=b} = \alpha b + \int_\alpha^5 f(x)\ dx$$

These are all the necessary components for making our calculation. What we need is:

$$\alpha b + \int_\alpha^5 f(x)\ dx = \frac{1}{2}\int_0^5 f(x) dx$$

One more notation change will make things a bit easier to deal with here, as well as to eliminate b from our expression: Note that b really is $f(\alpha)$, so we can say:

$$\alpha f(\alpha) + \int_\alpha^5 f(x)\ dx = \frac{1}{2}\int_0^5 f(x) dx$$

Now we are ready to solve. First, let's do some algebra on all of our terms:

$$\alpha f(\alpha) + \int_\alpha^5 f(x)\ dx = \frac{1}{2}\int_0^5 f(x) dx$$

$$\alpha f(\alpha) + \int_\alpha^5 f(x)\ dx = \frac{1}{2}\int_0^\alpha f(x)\ dx + \frac{1}{2}\int_\alpha^5 f(x)\ dx \quad \text{(split up integral on the right)}$$

$$\alpha f(\alpha) + \frac{1}{2}\int_\alpha^5 f(x)\ dx = \frac{1}{2}\int_0^\alpha f(x)\ dx \quad \text{(combine like terms, via integral properties)}$$

$$\alpha f(\alpha) + \frac{1}{2}\int_\alpha^5 f(x)\ dx - \frac{1}{2}\int_0^\alpha f(x)\ dx = 0 \quad \text{(subtract)}$$

That last subtraction can be done because we know our integrand is a polynomial, and the antiderivative of a polynomial is another polynomial, still. We can also predict that the polynomial we get will be in terms of α from our notation in the expression. Now we can proceed:

$$\alpha f(\alpha) + \frac{1}{2}\int_\alpha^5 f(x)\ dx - \frac{1}{2}\int_0^\alpha f(x)\ dx = 0$$

$$\alpha(25-\alpha^2)+\frac{1}{2}\int_{\alpha}^{5}(25-x^2)\ dx-\frac{1}{2}\int_{0}^{\alpha}(25-x^2)\ dx=0 \qquad \text{(substitute)}$$

$$\alpha(25-\alpha^2)+\frac{1}{2}\left[25x-\frac{1}{3}x^3\right]_{\alpha}^{5}-\frac{1}{2}\left[25x-\frac{1}{3}x^3\right]_{0}^{\alpha}=0 \qquad \text{(integrate)}$$

$$25\alpha-\alpha^3+\frac{1}{2}\left[\left(125-\frac{125}{3}\right)-\left(25\alpha-\frac{\alpha^3}{3}\right)\right]-\frac{1}{2}\left(25\alpha-\frac{\alpha^3}{3}\right)=0 \qquad \text{(evaluate antiderivatives at limits)}$$

$$25\alpha-\alpha^3+\frac{1}{2}\left(\frac{250}{3}-25\alpha+\frac{\alpha^3}{3}\right)-\frac{1}{2}\left(25\alpha-\frac{\alpha^3}{3}\right)=0 \qquad \text{(simplify)}$$

$$50\alpha-2\alpha^3+\left(\frac{250}{3}-25\alpha+\frac{\alpha^3}{3}\right)-\left(25\alpha-\frac{\alpha^3}{3}\right)=0 \qquad \text{(multiply through by 2)}$$

$$50\alpha-2\alpha^3+\frac{250}{3}-25\alpha+\frac{\alpha^3}{3}-25\alpha+\frac{\alpha^3}{3}=0 \qquad \text{(deal with parentheses)}$$

$$150\alpha-6\alpha^3+250-75\alpha+\alpha^3-75\alpha+\alpha^3=0 \qquad \text{(multiply through by 3)}$$

$$\begin{aligned} -4\alpha^3+250&=0 \qquad \text{(simplify)}\\ 2\alpha^3-125&=0\\ \alpha&=\sqrt[3]{\frac{125}{2}} \end{aligned}$$

The algebra was a little long, but it cleaned up quite nicely. Recall that $b=f(\alpha)$, so our last step is to say:

$$\begin{aligned} b&=f(\alpha)\\ &=f\left(\sqrt[3]{\frac{125}{2}}\right)\\ &=25-\left(\sqrt[3]{\frac{125}{2}}\right)^2\\ &=25-\left(\frac{125}{2}\right)^{\frac{2}{3}}\\ b&\approx 9.251 \end{aligned}$$

So, the line $y=25-\left(\frac{125}{2}\right)^{\frac{2}{3}}\approx 9.251$ cuts the area under the parabola in half.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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