Interesting series question - the notation saves us!

Phil Petrocelli, mymathteacheristerrible.com

August 25, 2022

Sometimes a simple change in perspective helps us solve a problem quicker. Here, a simple change in the original, somewhat daunting notation is just what we need.

Question.

For $n \epsilon \mathbb{N}$, what is the value of:

$$\sqrt[3]{\frac{(1\cdot 2\cdot 4)+(2\cdot 4\cdot 8)+...+(n\cdot 2n\cdot 4n)}{(1\cdot 3\cdot 9)+(2\cdot 6\cdot 18)+...+(n\cdot 3n\cdot 9n)}}$$

for $n = 2022$?

Solution.

This is one of those classic questions that we see in things like math contests. there are probably a variety of ways to do this, but I chose to do some simplifying and rewriting using different notation. Starting with the general term, we can rewrite $(n \cdot 2n \cdot 4n)$ as $8n^3$ by using the rules of exponents. Similarly, we can rewrite $(n \cdot 3n \cdot 9n)$ as $27n^3$, leaving us with

$$\sqrt[3]{\frac{(1\cdot 2\cdot 4)+(2\cdot 4\cdot 8)+...+(n\cdot 2n\cdot 4n)}{(1\cdot 3\cdot 9)+(2\cdot 6\cdot 18)+...+(n\cdot 3n\cdot 9n)}} = \sqrt[3]{\frac{(1\cdot 2\cdot 4)+(2\cdot 4\cdot 8)+...+8n^3}{(1\cdot 3\cdot 9)+(2\cdot 6\cdot 18)+...+27n^3}} \tag{1}$$

There is still a lot happening in the fraction under the radical! Notice that there are sums in both numerator and denominator, and recall that the sum in the denominator is one reason why cancelling would be very difficult. We *could* factor a 2 out of the top and a 3 out of the bottom, but that would not do us much good, it seems. (Try it and see.)

At this point, I noticed that we had a sum on top and bottom, and that the problem, as simplified in (1) gives us a *general term*. Once I realized that, I was able to rewrite our expression, yet again, using summation (sigma) notation:

$$\sqrt[3]{\frac{(1\cdot 2\cdot 4)+(2\cdot 4\cdot 8)+...+8n^3}{(1\cdot 3\cdot 9)+(2\cdot 6\cdot 18)+...+27n^3}} = \sqrt[3]{\frac{\sum\limits_{n=1}^{\infty} 8n^3}{\sum\limits_{n=1}^{\infty} 27n^3}}$$

The notation is cleaned up, sure, but some cool stuff is about to happen. Let's continue!

$$\sqrt[3]{\frac{\sum\limits_{n=1}^{\infty} 8n^3}{\sum\limits_{n=1}^{\infty} 27n^3}} = \sqrt[3]{\frac{8\sum\limits_{n=1}^{\infty} n^3}{27\sum\limits_{n=1}^{\infty} n^3}} \qquad \text{(factor constants from summations)}$$

$$= \sqrt[3]{\frac{8}{27}} \cdot \sqrt[3]{\frac{\sum_{n=1}^{\infty} n^3}{\sum_{n=1}^{\infty} n^3}} \qquad \text{(factor constants from cube roots)}$$

$$= \sqrt[3]{\frac{8}{27}} \cdot 1 \qquad \text{(cancel same sum in fraction!)}$$

$$= \frac{2}{3} \qquad \text{(simplify)}$$

Using the summation notation, for me, anyway, teased out that we would have $\sqrt[3]{1}$ by cancelling the same sum in the numerator and denominator. I am not sure I would have seen it if I had not condensed the original notation into the sigma notation. Once I did that, the problem was quickly solved. The notation saves us!

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

My email address is: phil.petrocelli@gmail.com.

Please visit `https://mymathteacheristerrible.com` for other study guides. Please tell others about it.

Please donate

I write these study guides with interest in good outcomes for math students and to be a part of the solution. If you would consider donating a few dollars to me so that these can remain free to everyone who wants them, please visit my PayPal and pay what you feel this is worth to you. Every little bit helps.

My PayPal URL is: `https://paypal.me/philpetrocelli`.

Thank you so much.