An infinite series problem, two ways

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Sometimes there are multiple ways to do a problem. This one can be approached somewhat systematically, with intuition helping us out.

Question.

Does the following series converge or diverge? Choose the appropriate test and use it:

$$S = \sum_{n=1}^{\infty} \frac{3}{\sqrt{5n-4}}$$

Solution.

First things first. Let's massage the problem statement to get at the essentials of the problem. This can be done by using the properties of summations to factor.

$$S = \sum_{n=1}^{\infty} \frac{3}{\sqrt{5n-4}}$$

$$S = 3\sum_{n=1}^{\infty} \frac{1}{\sqrt{5n-4}} \tag{1}$$

So, we really can look at this problem as asking if,

$$S_1 = \sum_{n=1}^{\infty} \frac{1}{\sqrt{5n-4}}$$

converges. The 3 really has no effect on convergence.

0 Always use the test for divergence first!

Remember, the test for divergence is: if $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. If it *does* equal zero, then the test is inconclusive, so another method must be used. Let's have a look. For this series, S_1:

$$a_n = \frac{1}{\sqrt{5n-4}}$$

and the test is:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{5n-4}} \qquad \text{(Test for divergence)}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{5n-4}} \qquad \text{(rule for radicals and fractions)}$$

$$= \sqrt{\lim_{n\to\infty} \frac{1}{5n-4}} \qquad \text{(limit rule for composite functions)}$$

So, for expressions such as these, involving a rational expression such as we have under the radical here, I tend to think of these using the leading coefficient test, which many of us learned around Algebra 2 or thereabouts. If we reach back into our memories, we will see that it's useful to think of the numerator of that fraction as $0n + 1$, allowing us to have an n term in the numerator with which to do the leading coefficient test. The test simply results in $\frac{0}{5}$, so:

$$\lim_{n\to\infty} a_n = \sqrt{\lim_{n\to\infty} \frac{1}{5n-4}}$$

$$= \sqrt{\frac{0}{5}} \qquad \text{(use leading coeff test to calculate limit)}$$

$$= \sqrt{0}$$

$$= 0$$

As stated, when the limit is zero, then we cannot conclude anything about convergence nor divergence. On to another test.

1 The Integral Test

My readers know I love integration and integration tricks. I make any excuse to solve an integral, and this is no exception. It turns out that this one is very straightforward with an elementary u-substitution.

Recall that if

$$\int_1^\infty f(x)\ dx$$

converges, then so does our series. Recall that $f(x)$ is a function we create from a_n, simply by replacing n with x. For our series, our $f(x)$ is:

$$f(x) = \frac{1}{\sqrt{5x-4}}$$

So, our integral is:

$$\int_1^\infty \frac{1}{\sqrt{5x-4}}\ dx$$

and we can begin with a u-substitution:

$$u = 5x - 4$$
$$du = 5\ dx$$

Having a look at the bounds of our integral, the bounds do not change, if we use the expression for u, above. Now we can proceed with solving our integral, which becomes:

$$\frac{1}{5}\int_1^\infty \frac{1}{\sqrt{u}}\,du = \frac{2}{5}\sqrt{u}\bigg]_1^\infty \qquad \text{(integrate)}$$

$$= \frac{2}{5}\left(\lim_{u\to\infty}\sqrt{u} - \sqrt{1}\right) \qquad \text{(substitute, create limit for improper upper bound)}$$

$$= \frac{2}{5}(\infty - 1) \qquad \text{(calculate limit, simplify)}$$

$$= \infty$$

Our integral diverges, and, therefore, so does our series.

2 The Limit Comparison Test

There is a second way we can get this result: the Limit Comparison Test. Recall that the essence of this comparison test is to choose a known convergent or divergent series, make a fraction with their general terms, and see what happens as $n \to \infty$. If the result of taking the limit is greater than zero, then our series has the same behavior as the series we're comparing it to: diverges if compared to a known divergent, converges if compared to a known convergent.

To set this up in symbols, we have our a_n term already and the one we will compare with is b_n. And the limit is:

$$L = \lim_{n\to\infty}\frac{a_n}{b_n}$$

The tricky part of using this test is choosing a known convergent or divergent, the b_n term. In this case, we can use the series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

This makes our b_n term:

$$b_n = \frac{1}{\sqrt{n}}$$

If we rewrite the b_n term as:

$$b_n = \frac{1}{n^{\frac{1}{2}}}$$

we can see that this general term is one of a *p-series* and since $p = \frac{1}{2} \leq 1$, this is a divergent p-series. If our limit is greater than zero in the comparison test, our given series will also diverge:

$$L = \lim_{n\to\infty}\frac{a_n}{b_n}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{\sqrt{5n-4}}}{\frac{1}{n^{\frac{1}{2}}}} \qquad \text{(substitute)}$$

$$= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{\sqrt{5n-4}} \qquad \text{(clean up compound fraction)}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{5n-4}} \qquad \text{(change notation for the numerator)}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n}{5n-4}} \qquad \text{(rule for radicals in fractions)}$$

$$= \sqrt{\lim_{n\to\infty} \frac{n}{5n-4}} \qquad \text{(limit rule for composite functions)}$$

$$L = \frac{1}{\sqrt{5}} \qquad \text{(leading coeff test for the limit, then simplify)}$$

Per our discussion, since $\frac{1}{\sqrt{5}} > 0$, then our series diverges, since we compared it to a known divergent.

Done.

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I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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