An interesting similar triangles problem

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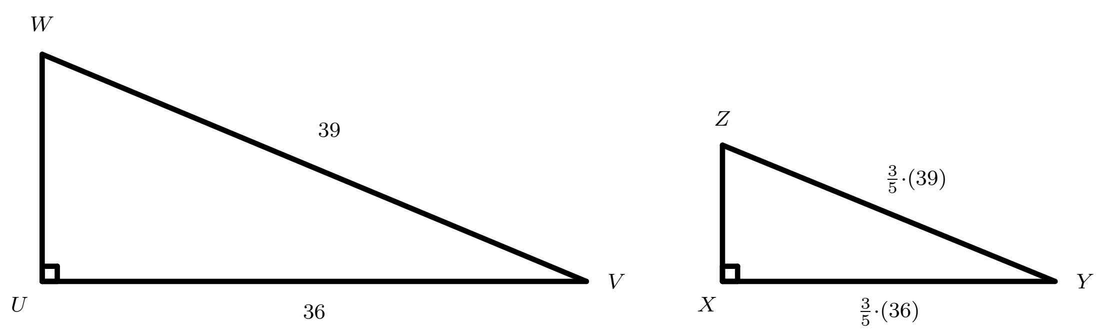
I did this problem with a student as part of SAT coaching. Using a little insight, we can make a problem that looks to have difficult calculations a little easier.

Question.

In triangle UVW, the measure of $\angle U$ is 90°, $WV = 39$, and $UV = 36$. Triangle XYZ is similar to triangle UVW, where $\angle X$, $\angle Y$, and $\angle Z$ correspond to $\angle U$, $\angle V$, and $\angle W$, respectively. If each side of triangle XYZ is $\frac{3}{5}$ the length of its corresponding side of triangle UVW, what is the value of $cos\ Z$?

Solution.

Let's start with a diagram of the situation:



One important detail here is that this problem appeared in the *no calculator* math section of a practice test. The reason that it's important is that it appears that using the Pythagorean Theorem would eat up precious time on a timed exam; we want to avoid wasting time with long calculations as much as possible. There's no way we want to square 39 and 36 and do *that* math, unless it's the only possible way.

The real issue here is that we need that vertical side's length to calculate the cosine of W and Z. I think I started thinking about an identity like:

$$cos\theta = \frac{1}{2} \cdot \frac{sin2\theta}{sin\theta}$$

or the cofunction identity that we use in right triangles all the time:

$$cos\theta = sin(90^\circ - \theta) \qquad \text{where } \theta \text{ is one acute angle and } (90^\circ - \theta) \text{ is the other acute angle}$$

but both require the vertical side to calculate.

Out of curiosity, I decided I would calculate what $cosV$ was, since we had both of the needed values:

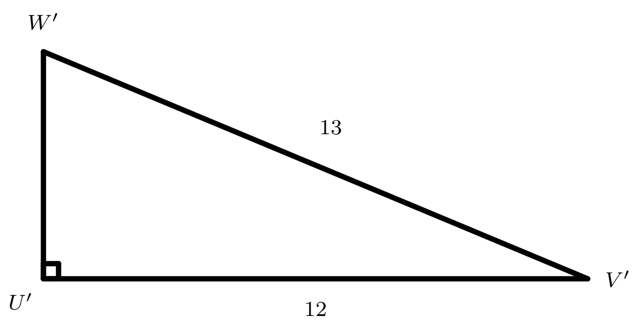
$$cosV = \frac{\text{adj}}{\text{hyp}}$$

$$= \frac{36}{39}$$

$$= \frac{12}{13} \qquad \text{(reduce)}$$

Doesn't appear useful... Or, is it?[1] Turns out, it is. Since we're dealing with similar triangles, we are dealing with two things: corresponding angles that are equal and proportions of the sides - to each other. The first ensures 180 total degrees for the 3 angles, the latter ensures a unique triangle given those 3 angles.

Since the base of either triangle has to be related to the hypoteneuse of the triangle in a ratio of $\frac{12}{13}$, we can create a new triangle - a *smaller* triangle - $U'V'W'$, whose base and hypotenuse are 12 and 13, respectively:



Suddenly, *way* more tractable! We remember that $13^2 = 169$ and that $12^2 = 144$, at the very least, and we can calculate that the remaining side is 5. Perhaps you've even seen another common Pythagorean triple besides $(3, 4, 5)$? One of them is $(5, 12, 13)$! Remembering that Pythagorean triple is the fastest way to solution: no squaring, no square roots needed.

Now, our cosine calculation is speedy: $cosZ = cosW' = \frac{5}{13}$, where Z and W' are corresponding angles of our similar triangles.

Done.

[1] Credit to VSauce, `https://youtube.com/vsauce`

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I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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