A system of exponential and logarithmic equations

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Systems of *linear* equations are part of high school Mathematics studies. But what about *other* types of systems of equations? Here, we'll look at a system that mixes exponential and logarithmic equations.

Question.

Find x and y:

$$log_3(x+y) + log_3(x-y) = 1 \tag{1}$$

$$4^{\frac{x}{y}+\frac{y}{x}} = 32 \tag{2}$$

Solution.

This likely looks unfamiliar. Unfamiliarity is a bit of a signal for us to try to make sense of these two equations, to get them kind of looking the same, such that we can use basic techniques for systems (elimination, substitution, graphing - same for all systems of equations) to solve.

It is useful to note that exponents and logs are inverses of each other, so converting at least one of the equations to its complementary form using

$$b^a = x \;\equiv\; log_b x = a \tag{3}$$

seems like a good idea.

1 Domains

It's worth it to talk about domains before we start. Note that in equation (1), we have logarithms, and recall that the domain of $y = log(x)$ is $x > 0$. For equation (1), therefore, we have the following inequalities for the arguments of both of the log terms:

$$x + y > 0$$
$$x - y > 0$$

Having a look at equation (2), we can exponentiate anything, but since our exponent is a sum of two fractions, we have the additional domain constraints of:

$$x \neq 0$$
$$y \neq 0$$

These restrictions on the domains of our equations will inform our choices of solutions as we complete the calculations here.

2 The log equation, (1)

The log terms having the same base - 3 - and the fact that they're added together is a signal to use this log property that will allow us to condense the equation:

$$log(a \cdot b) = log(a) + log(b)$$

Condensing (1), we have:

$$\begin{aligned}
log_3(x+y) + log_3(x-y) &= 1 \\
log_3[(x+y)(x-y)] &= 1 && \text{(condense log expression)} \\
log_3(x^2 - y^2) &= 1 && \text{(expand, using difference of squares pattern)} \\
x^2 - y^2 &= 3^1 && \text{(convert to exponent form, using (3))}
\end{aligned}$$

Not bad. On to the exponential equation:

3 The exponential equation, (2)

Recall that exponential equations are much easier to solve when here is a common base among the exponential terms. Fortunately for us, both 4 and 32 are powers of 2 - 2^2 and 2^5, respectively. Note that we also have a sum of fractions in the exponent that we should add together to form one exponent: $\frac{x}{y} + \frac{y}{x} = \frac{x^2+y^2}{xy}$. Let's work these into equation (2):

$$\begin{aligned}
4^{\frac{x}{y}+\frac{y}{x}} &= 32 \\
(2^2)^{\frac{x}{y}+\frac{y}{x}} &= 2^5 && \text{(convert bases to powers of 2)} \\
2^{2(\frac{x}{y}+\frac{y}{x})} &= 2^5 && \text{(power to a power rule: multiply)} \\
2^{2\left(\frac{x^2+y^2}{xy}\right)} &= 2^5 && \text{(substitute combined fraction in exponent)} \\
2^{\frac{2x^2+2y^2}{xy}} &= 2^5 && \text{(distribute the 2 to the fractional exponent)}
\end{aligned}$$

There is a lot going on, but not too bad. One final step is to simplify this question by taking the log_2 of both sides:

$$\begin{aligned}
2^{\frac{2x^2+2y^2}{xy}} &= 2^5 \\
log_2\left[2^{\frac{2x^2+2y^2}{xy}}\right] &= log_2[2^5] && \text{(take the log of both sides)} \\
\frac{2x^2+2y^2}{xy} &= 5 && \text{(keep the exponents)} \\
\frac{x^2+y^2}{xy} &= \frac{5}{2} && \text{(divide by 2)}
\end{aligned}$$

We can continue to work on this equation a bit more, and we will find that we will have a quadratic:

$$\frac{x^2+y^2}{xy} = \frac{5}{2}$$

$$x^2+y^2 = \frac{5}{2}xy \qquad \text{(since neither } x \text{ nor } y \text{ is zero, we can multiply by } xy\text{)}$$

$$x^2 - \frac{5}{2}xy + y^2 = 0 \qquad \text{(standard form quadratic!)}$$

$$2x^2 - 5xy + 2y^2 = 0 \qquad \text{(clear the fraction by multiplying by 2)}$$

And since we have a standard form quadratic, we can solve using the usual techniques: factoring, completing the square, or quadratic formula. There is one trick here, however. If we take this to be *a quadratic in (independent variable)* x, then any y terms can be thought of as constants. Let's use the quadratic formula to see what I mean. For a quadratic in x:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

for our quadratic in x: $2x^2 - 5xy + 2y^2 = 0$, we have

$$a = 2$$

$$b = -5y \qquad (y \text{ is a constant})$$

$$c = 2y^2 \qquad (y \text{ is a constant})$$

With this, we're ready to plug into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(-5y) \pm \sqrt{(-5y)^2 - 4\cdot 2\cdot 2y^2}}{2\cdot 2}$$

$$= \frac{5y \pm \sqrt{25y^2 - 16y^2}}{4}$$

$$= \frac{5y \pm \sqrt{9y^2}}{4}$$

$$= \frac{5y \pm 3y}{4}$$

This, of course, leaves us with two solutions:

$$x = \frac{5y+3y}{4} \qquad \text{(take the + sign)}$$

$$x = \frac{8y}{4} \qquad \text{(combine like terms)}$$

$$x = 2y \qquad \text{(simplify)}$$

and

$$x = \frac{5y - 3y}{4} \qquad \text{(take the - sign)}$$

$$x = \frac{2y}{4} \qquad \text{(combine like terms)}$$

$$x = \frac{1}{2}y \qquad \text{(simplify)}$$

4 Tying it all together: Using substitution with equation (1)

We are nearing solution! We have two values for x from the last section, and our remaining equation:

$$x^2 - y^2 = 3$$

Let's apply our two x-values to the remaining equation to complete solving this system.

For $x = 2y$:

$$\begin{aligned} x^2 - y^2 &= 3 \\ (2y)^2 - y^2 &= 3 && \text{(substitute)} \\ 4y^2 - y^2 &= 3 && \text{(square the term in parentheses)} \\ 3y^2 &= 3 && \text{(combine like terms)} \\ y &= \pm 1 && \text{(divide by 3, then take square root)} \end{aligned}$$

Since $x = 2y$, his result leaves us with two points: $(-2, -1)$ and $(2, 1)$. $(-2, -1)$ violates our constraint of $x+y > 0$, so we throw it out; only $(2, 1)$ remains.

Similarly, for $x = \frac{1}{2}y$:

$$\begin{aligned} x^2 - y^2 &= 3 \\ \left(\frac{1}{2}y\right)^2 - y^2 &= 3 && \text{(substitute)} \\ \frac{1}{4}y^2 - y^2 &= 3 && \text{(square term in parentheses)} \\ -\frac{3}{4}y^2 &= 3 && \text{(combine like terms)} \\ y^2 &= -4 \end{aligned}$$

This has no real solutions (can't take the square root of a negative number), so the answer to this system of equations is $(x, y) = (2, 1)$.

A graph of this system can be found in this Desmos workbook.

Done.

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