The Ellipse-Hyperbola Connection: Graphing made easy!

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My students know that I am not crazy about studying Conic Sections. It always feels like too much stuff to manage and too much rote memorization. A recent student of mine pointed out the ellipse-hyperbola connection and made it way more specific than a previous connection I noticed.

Consider.

The standard form of an ellipse is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{1}$$

The standard form of a hyperbola is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{2}$$

Is there a connection between them that we can exploit to make describing and graphing them easier?

Discussion.

My students and readers all know that my favorite tool for discussion and exploration is Desmos (`http://www.desmos.com`) so it should be no surprise that I created a workbook to explore this topic: `https://www.desmos.com/calculator/bmhg2q`

What I did here was to choose some parameters for simplicity's sake, namely that the center of the ellipse and hyperbola is $(0,0)$. In order to explore these two conics in all ways (center, excepted), I added two sliders for a and b. Side some sliders around and get comfortable with the workbook and convince yourself that it works.

The key thing to notice in (1) and (2) is that the equations are identical, except for the sign in between the x and y terms: + means we have an ellipse, minus means we have a hyperbola. One device I used to keep these standard forms straight was to think of the ellipse - the one with the + sign - as *uniting* the halves of the figure, as in: **()**. And, the other - the one with the minus sign - as *repelling* both halves of the figure, as in: **)(**. This really helps when determining what shape we're dealing with.

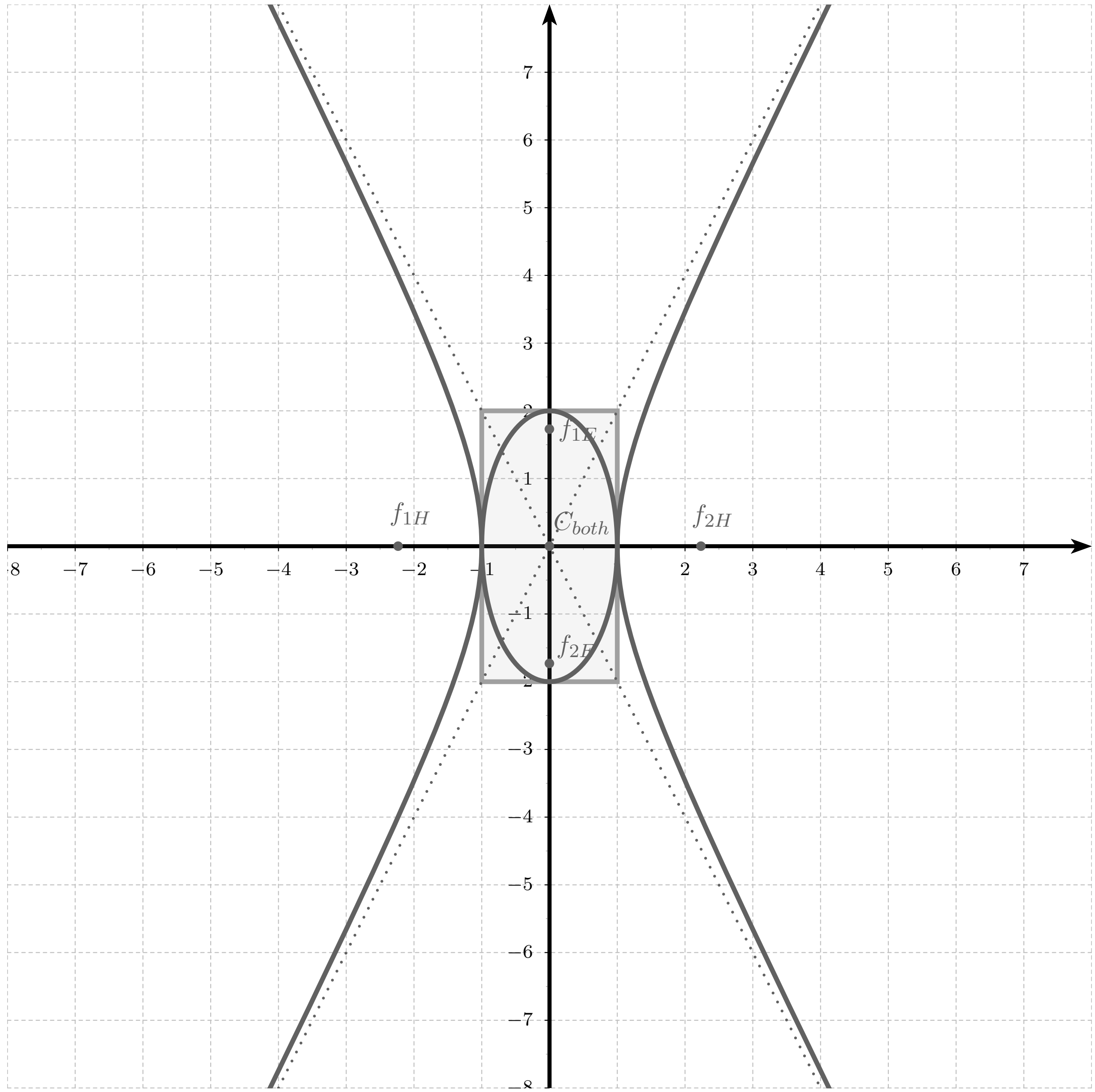
So, a simple problem solving strategy we can adopt is, **first, draw the ellipse, then use it to draw the hyperbola**. Let's take a simple example.

For simplicity, let ellipse E and hyperbola H be defined as:

$$\text{E:} \qquad \frac{x^2}{1^2} + \frac{y^2}{2^2} = 1 \tag{3}$$

$$\text{H:} \qquad \frac{x^2}{1^2} - \frac{y^2}{2^2} = 1 \tag{4}$$

Let's have a look at the graph:



If we start with the equation of E, we can see that the major axis has to be $2a$ units long, and we know $a = 2$ from the equation. Similarly, we know the minor axis has to be $2b$, since $b = 1$ from the equation. And, in our simple case, the center of E is the origin, $(0, 0)$. The major axis has to be parallel to the y-axis, since a is under the y term in the equation. This is quite enough to draw the ellipse, E.

Now, we are ready to draw H, the hyperbola. The key starting point is to first draw a rectangle that encloses the ellipse. Please note from the diagram, that both the ellipse and the hyperbola have the same center, C. And! The vertices of the hyperbola are the endpoints of the *minor* axis on the ellipse. If you remember the $() \rightleftarrows)($ analogy, you can really see it play out on this graph. Please also note that the corners of the rectangle serve as points on the asymptotes, where another point is the center of the ellipse: calculate the slope by using these two points and then use the point-slope formula[1] to get the equations of the lines. The asymptotes and vertices are enough to be able to sketch the hyperbola! It's so much easier!

To be able to find the foci, f_{1H} and f_{2H}, use: $c^2 = a^2 + b^2$. The foci are found $\pm c$ units from the center, C, along the minor axis of the ellipse.

[1] $y - y_1 = m(x - x_1)$, given $P(x_1, y_1)$ and slope m.

Please do play around with `https://www.desmos.com/calculator/bmhg2qdec0` to see that the situation is exactly the same, no matter the orientation of the ellipse and hyperbola. The relationships discussed in this article always hold.

Done.

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

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Thank you so much.