Flex your trig identity muscles with this tricky trig limit!

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The use of trigonometric identities to rephrase problems in Calculus is a very important skill. In this limits problem, it really really helps!

Question.

Calculate:

$$L = \lim_{x \to 0} \frac{x - x \cdot cosx}{sin^2 3x}$$

Solution.

Anytime we see a limit involving trigonometric functions as x approaches zero, we should recall two basic trigonometric limits:

$$\lim_{x \to 0} \frac{sinx}{x} = 1 \tag{1}$$

and

$$\lim_{x \to 0} \frac{cosx - 1}{x} = 0 \tag{2}$$

A great approach in dealing with trigonometric limits is to try to rephrase the given limit, L, in terms of these two limits, if possible.

For rational functions, it's a good idea to factor numerator and denominator:

$$L = \lim_{x \to 0} \frac{x - x \cdot cosx}{sin^2 3x}$$

$$= \lim_{x \to 0} \frac{x(1 - cosx)}{sin^2 3x} \qquad \text{(factor)}$$

Now, whenever I see an expression like $(1 - cosx)$ mixed with powers of $sinx$ as we have in the denominator, I try to make use of difference of squares via multiplying by the conjugate of the expression. This is so that I can make use of the Pythagorean identities for sine and cosine:

$$L = \lim_{x \to 0} \frac{x(1 - cosx)}{sin^2 3x}$$

$$= \lim_{x \to 0} \frac{x(1 - cosx)}{sin^2 3x} \cdot \frac{(1 + cosx)}{(1 + cosx)} \qquad \text{(multiply top \& bottom by conjugate)}$$

$$= \lim_{x \to 0} \frac{x(1 - cos^2x)}{sin^2 3x \cdot (1 + cosx)} \qquad \text{(multiply out the top using diff of squares pattern)}$$

$$= \lim_{x \to 0} \frac{x \cdot sin^2x}{sin^2 3x \cdot (1 + cosx)} \qquad \text{(simplify numerator using Pythagorean identity)}$$

It *does* look like a mess, but here is where some intuitive work comes in. Since there are a bunch of products in the numerator and denominator, we can start to pull things apart so as to strategize:

$$L = \lim_{x \to 0} \frac{x \cdot sin^2x}{sin^2 3x \cdot (1 + cosx)}$$

$$= \lim_{x \to 0} \frac{x}{sin3x} \cdot \frac{sin^2x}{sin3x \cdot (1 + cosx)} \qquad \text{(pull apart fractions, strategically, so we can use (1))}$$

Now, we can employ the product rule for limits:

$$L = \lim_{x \to 0} \frac{x}{sin3x} \cdot \frac{sin^2x}{sin3x \cdot (1 + cosx)}$$

$$= \lim_{x \to 0} \frac{x}{sin3x} \cdot \lim_{x \to 0} \frac{sin^2x}{sin3x \cdot (1 + cosx)}$$

Looking at the first limit in our product, we see that the numerator doesn't *quite* match the expression for the angle in the sine term in the denominator. This is easily fixed if we multiply the top and bottom of that fraction by 3:

$$L = \lim_{x \to 0} \frac{x}{sin3x} \cdot \lim_{x \to 0} \frac{sin^2x}{sin3x \cdot (1 + cosx)}$$

$$= \lim_{x \to 0} \frac{3x}{3sin3x} \cdot \lim_{x \to 0} \frac{sin^2x}{sin3x \cdot (1 + cosx)} \qquad \text{(mult top and bottom by 3)}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{3x}{sin3x} \cdot \lim_{x \to 0} \frac{sin^2x}{sin3x \cdot (1 + cosx)} \qquad \text{(factor 3 from the denominator, now 3x in both places)}$$

$$= \frac{1}{3} \cdot \lim_{x \to 0} \frac{sin^2x}{sin3x \cdot (1 + cosx)} \qquad \text{(apply (1) - it works for the reciprocal, too)}$$

Not too bad! We are down to one limit now. Looking ahead to evaluating the remaining limit, we see that the $sin3x$ term in the denominator is going to be problematic, since it yields zero when we plug in zero for x. Recall the double angle formula for sine:

$$sin2x = 2\ sinx\ cosx$$

Note that if this was the case in our denominator, the $sinx$ produced by the double angle formula would cancel one of the $sinx$ terms in the numerator, and we would be done. But! We don't have a double angle for the input to sine, rather, we have a *triple* angle. But what the heck is the triple angle formula? I can't be bothered memorizing it, but I can always reproduce it using a trick and some nimble trig identity manipulations. If we think of $3x$ as $(2x + x)$, we can use the angle sum identity for sine:

$$
\begin{aligned}
sin3x &= sin(2x + x) \\
&= sin2x \cdot cosx + sinx \cdot cos2x && \text{(apply angle sum identity for sine)} \\
&= 2sinx \cos^2 x + sinx(cos^2x - sin^2x) && \text{(expand double angle expressions)} \\
&= sinx(2cos^2x + cos^2x - sin^2x) && \text{(factor)} \\
&= sinx(3cos^2x - 1 + cos^2x) && \text{(simplify using Pythagorean identities)} \\
&= sinx(4cos^2x - 1) && \text{(simplify)}
\end{aligned}
$$

And there we have it. Now, let's substitute into our limit expression and solve:

$$
\begin{aligned}
L &= \frac{1}{3} \cdot \lim_{x \to 0} \frac{sin^2x}{sin3x \cdot (1 + cosx)} \\
&= \frac{1}{3} \cdot \lim_{x \to 0} \frac{sin^2x}{sinx(4cos^2x - 1)(1 + cosx)} && \text{(substitute)} \\
&= \frac{1}{3} \cdot \lim_{x \to 0} \frac{sinx}{(4cos^2x - 1)(1 + cosx)} && \text{(cancel sine term)} \\
&= \frac{1}{3} \cdot \frac{0}{3 \cdot 2} && \text{(evaluate the limit)} \\
&= 0 && \text{(simplify)}
\end{aligned}
$$

Done.

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