Same Integral, Two Ways to Solve, Redux

Phil Petrocelli, mymathteacheristerrible.com

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As I mentioned in a previous article, there are many ways to deal with integration problems. As with the last one, I also got this one from Reddit. I'll discuss two ways to do it.

Question.

Integrate the following:

$$\int tan4x\ sec^4 4x\ dx$$

.

Solution, using powers of trig functions technique.

So, every calculus book covers how to deal with combinations of trig functions multiplied in an integrand. The classic pairings are *sinx* and *cosx*, *secx* and *tanx*, and *cscx* and *cotx*. We can see that this integral is of the *tanx* and *secx* variety. Let's prepare by rewriting and making a u-substitution.

$$\begin{aligned} \int tan4x\ sec^4 4x\ dx &= \int tan4x\ sec4x\ sec^3 x\ dx && \text{(separate out one } sec4x\text{)} \\ &= \int (sec4x\ tan4x)\ sec^3 x\ dx && \text{(rewrite, clarify with parens)} \end{aligned}$$

Now, let:

$$\begin{aligned} u &= sec4x \\ du &= 4sec4x\ tan4x\ dx \\ dx &= \frac{du}{4sec4x\ tan4x} \end{aligned}$$

Then:

$$\int (sec4x\ tan4x)\ sec^3x\ dx = \int sec4x\ tan4x\ u^3\ \frac{du}{4sec4x\ tan4x} \qquad \text{(substitute)}$$

$$= \frac{1}{4}\int u^3\ du \qquad \text{(simplify)}$$

$$= \frac{1}{4}\cdot\frac{1}{4}u^4 \qquad \text{(integrate)}$$

$$= \frac{1}{16}sec^4 4x + C$$

So:

$$\int tan4x\ sec^4 4x\ dx = \frac{1}{16}sec^4 4x + C$$

Done.

Solution, using an alternate u-substitution.

We can do our substitutions another way, however. This one is kind of interesting. We start by rewriting and doing a different u-substitution:

$$\int tan4x\ sec^4 4x\ dx = \int tan4x\ (sec^2 4x)^2\ dx \qquad \text{(rewrite)}$$

Now, let:

$$u = tan4x$$
$$du = 4sec^2 4x\ dx$$
$$dx = \frac{du}{4sec^2 4x}$$

Then:

$$\int tan4x\ (sec^2 4x)^2\ dx = \int u\ (sec^2 4x)^2 \frac{du}{4sec^2 4x} \qquad \text{(substitute)}$$

$$= \frac{1}{4}\int sec^2 4x\ u\ du$$

Seems like we have too many $sec4x$ terms! But:

$$tan^2\theta + 1 = sec^2\theta$$

so we can say:

$$tan^2 4x + 1 = sec^2 4x$$

and, since $u = tan4x$, we can say:

$$u^2 + 1 = sec^2 4x$$

Making our integral:

$$\frac{1}{4}\int sec^2 4x\ u\ du = \frac{1}{4}\int u(u^2+1)\ du \qquad \text{(simplify)}$$

$$= \frac{1}{4}\int (u^3+u)\ du \qquad \text{(expand integrand)}$$

$$= \frac{1}{4}\left(\frac{1}{4}u^4 + \frac{1}{2}u^2\right) \qquad \text{(integrate)}$$

$$= \frac{1}{4}\left(\frac{tan^4 4x}{4} + \frac{tan^2 4x}{2}\right) + C$$

So:

$$\int tan4x\ sec^4 4x\ dx = \frac{1}{4}\left(\frac{tan^4 4x}{4} + \frac{tan^2 4x}{2}\right) + C$$

Done.

Ok. These two answers look different. Let's consider the case when both constants of integration (C) equal zero, for simplicity. If these answers are the same, as far as computing an indefinite integral goes, they should be equal or off by a constant. Is this true for our two answers?

If we look at the graphs for both answers, they're not exactly the same. Have a look: `https://www.desmos.com/calculator/w6pzhxy0mf`

The graphs look slightly off. Can we show that they're the same, say, by doing a proof? Yes.

$$\frac{1}{16}sec^4 4x \stackrel{?}{=} \frac{1}{4}\left(\frac{tan^4 4x}{4} + \frac{tan^2 4x}{2}\right)$$

$$\stackrel{?}{=} \frac{1}{16}(tan^4 4x + 2tan^2 4x) \qquad \text{(factor)}$$

$$\stackrel{?}{=} \frac{1}{16}tan^2 4x(tan^2 4x + 2) \qquad \text{(factor)}$$

$$\stackrel{?}{=} \frac{1}{16}(sec^2 4x - 1)(sec^2 4x + 1) \qquad \text{(use Pythagorean identities! Slick!)}$$

$$\stackrel{?}{=} \frac{1}{16}(sec^4 4x - 1) \qquad \text{(expand)}$$

$$\stackrel{?}{=} \frac{1}{16}sec^4 4x - \frac{1}{16} \qquad \text{(distribute)}$$

$$\frac{1}{16}sec^4 4x \stackrel{?}{=} \frac{1}{16}sec^4 4x - \frac{1}{16}$$

Well. We see that the answers are not exactly equal, but they *do* differ by a constant ($\frac{1}{16}$), and being off by a constant in this way indicates equality of indefinitely integrating the original integrand. Pretty neat! If you go back to the graph and subtract $\frac{1}{16}$ from $\frac{1}{16}sec^4 4x$, the graphs line up exactly!

Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

My email address is: phil.petrocelli@gmail.com.

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Thank you so much.