Trig integral two ways: Something interesting happens!

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Seeing multiple ways to solve a problem is one of the beautiful things about using the tools of Mathematics in creative ways. This integral, done two different ways, reveals something interesting about two inverse trig functions.

Question.

Integrate:

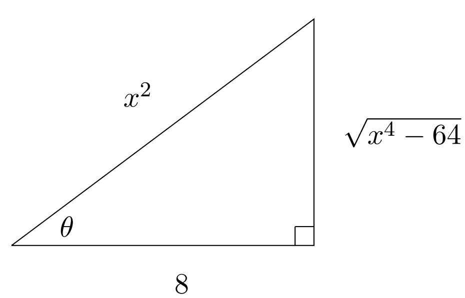
$$\int \frac{1}{x\sqrt{x^4 - 64}}\, dx$$

Solution.

The experienced Calculus student will recognize that this appears to be a trigonometric substitution situation. Indeed, this is true, in one sense. Here are two ways to approach this.

1 First way: Trig substitution

The first important thing to note here is that we can consider x^4 to be $(x^2)^2$. Consider a right triangle labeled as such:



Using the rules of trig substitution, we let:

$$sec\theta = \frac{x^2}{8}$$

$$8sec\theta = x^2 \qquad \text{(rearrange, this will be useful later)}$$

Taking a derivative, we have:

$$8sec\theta tan\theta d\theta = 2x\ dx$$

$$\frac{4sec\theta tan\theta}{x} d\theta = dx \qquad \text{(solve for } dx\text{)}$$

Substituting into the original integral, we have:

$$\int \frac{1}{x\sqrt{x^4 - 64}}\ dx = \int \frac{1}{x\sqrt{(8sec\theta)^2 - 64}} \cdot \frac{4sec\theta tan\theta}{x}\ d\theta \qquad \text{(substitute)}$$

$$= \int \frac{4sec\theta tan\theta}{x^2\sqrt{(8sec\theta)^2 - 64}}\ d\theta \qquad \text{(rearrange)}$$

$$= \int \frac{4sec\theta tan\theta}{8sec\theta\sqrt{(8sec\theta)^2 - 64}}\ d\theta \qquad \text{(substitute useful } x^2 \text{ term)}$$

$$= \frac{1}{2}\int \frac{\cancel{sec\theta} tan\theta}{\cancel{sec\theta}\sqrt{(8sec\theta)^2 - 64}}\ d\theta \qquad \text{(simplify)}$$

$$= \frac{1}{2}\int \frac{tan\theta}{\sqrt{64sec^2\theta - 64}}\ d\theta \qquad \text{(expand under radical in denominator)}$$

$$= \frac{1}{2}\int \frac{tan\theta}{8\sqrt{sec^2\theta - 1}}\ d\theta \qquad \text{(factor denominator)}$$

$$= \frac{1}{16}\int \frac{tan\theta}{\sqrt{tan^2\theta}}\ d\theta \qquad \text{(simplify denominator)}$$

$$= \frac{1}{16}\int \frac{\cancel{tan\theta}}{\cancel{\sqrt{tan^2\theta}}}\ d\theta \qquad \text{(the trig sub is working!)}$$

$$= \frac{1}{16}\int d\theta \qquad \text{(simplify)}$$

$$= \frac{1}{16}\theta + C \qquad \text{(integrate)}$$

Undoing our substitution, first solving for θ:

$$sec\theta = \frac{x^2}{8}$$

$$sec^{-1}sec\theta = sec^{-1}\left(\frac{x^2}{8}\right) \qquad \text{(inverse } sec \text{ of both sides)}$$

$$\cancel{sec^{-1}}\cancel{sec}\theta = sec^{-1}\left(\frac{x^2}{8}\right) \qquad \text{(cancel)}$$

$$\theta = sec^{-1}\left(\frac{x^2}{8}\right) \qquad \text{(simplify)}$$

So, our solution is:

$$\int \frac{1}{x\sqrt{x^4-64}}\,dx = \frac{1}{16}sec^{-1}\left(\frac{x^2}{8}\right) + C$$

2 Second way: u-substitution that leads to trig integral

Starting with our original integral, let $u = \sqrt{x^4-64}$ and perform a u-substitution:

$$u = \sqrt{x^4-64}$$

$$x^4 = u^2 + 64 \qquad \text{(rearrange, this will be useful later)}$$

Taking a derivative, we have:

$$du = \frac{1}{2\sqrt{x^4-64}} \cdot 4x^3\,dx$$

$$dx = \frac{\sqrt{x^4-64}\,du}{2x^3} \qquad \text{(solve for } dx\text{)}$$

Substituting into the original integral, we have:

$$\int \frac{1}{x\sqrt{x^4-64}}\,dx = \int \frac{\sqrt{x^4-64}\,du}{2x^3 \cdot x\sqrt{x^4-64}} \qquad \text{(substitute)}$$

$$= \int \frac{\cancel{\sqrt{x^4-64}}\,du}{2x^3 \cdot x\cancel{\sqrt{x^4-64}}} \qquad \text{(cancel)}$$

$$= \frac{1}{2}\int \frac{du}{x^4} \qquad \text{(simplify)}$$

$$= \frac{1}{2}\int \frac{du}{u^2+64} \qquad \text{(substitute useful } x^4 \text{ term)}$$

$$= \frac{1}{2}\int \frac{du}{u^2+8^2} \qquad \text{(simplify denominator)}$$

$$= \frac{1}{2}\int \frac{du}{8^2 \cdot [(\frac{u}{8})^2+1]} \qquad \text{(factor denomnator)}$$

$$= \frac{1}{2 \cdot 8^2}\int \frac{du}{[(\frac{u}{8})^2+1]} \qquad \text{(simplify)}$$

$$= \frac{1}{128} \int \frac{du}{[(\frac{u}{8})^2 + 1]} \qquad \text{(simplify)}$$

To deal with this *arctan* integral, we can make another substitution:

$$v = \frac{u}{8}$$

$$dv = \frac{1}{8} du \qquad \text{(differentiate)}$$

$$du = 8dv \qquad \text{(solve for } du\text{)}$$

Substituting:

$$\frac{1}{128} \int \frac{du}{[(\frac{u}{8})^2 + 1]} = \frac{1}{128} \int \frac{8dv}{v^2 + 1}$$

$$= \frac{1}{16} \int \frac{dv}{v^2 + 1} \qquad \text{(factor and simplify)}$$

$$= \frac{1}{16} tan^1 v + C \qquad \text{(integrate)}$$

Undoing the v and u substitutions, we get:

$$\frac{1}{16} tan^1 v + C = \frac{1}{16} tan^{-1} \frac{u}{8} + C \qquad \text{(undoing } v\text{, going back to } u\text{)}$$

$$= \frac{1}{16} tan^{-1} \frac{\sqrt{x^4 - 64}}{8} + C \qquad \text{(undoing } u\text{, going back to } x\text{)}$$

So, our solution is:

$$\int \frac{1}{x\sqrt{x^4 - 64}} \, dx = \frac{1}{16} tan^{-1} \frac{\sqrt{x^4 - 64}}{8} + C$$

3 Wait. These two solutions don't look the same.

We got two answers for the given integral:

$$y = \frac{1}{16} sec^{-1} \left(\frac{x^2}{8} \right) + C$$

and

$$y = \frac{1}{16} tan^{-1} \frac{\sqrt{x^4 - 64}}{8} + C$$

The easiest way to see that these are the same is to check out the graphs in Desmos.[1]

A more interesting way to see that these are the same is by doing some algebra that can lead us to a simple proof. Let's have a look.

An easy way to do this - with a proof mindset - is to manipulate each expression for y separately and see if we can get them to look like each other. The thing that makes this the toughest, to me, is the inverse trig functions in both expressions. If we could isolate the inverse trig functions and take the inverse of those inverse functions, then the world of trig identities takes center stage as the main path to solution.[2] Let's start with the first expression for y that we got from the trig sub method:

$$y = \frac{1}{16} sec^{-1}\left(\frac{x^2}{8}\right) + C$$

$$y - C = \frac{1}{16} sec^{-1}\left(\frac{x^2}{8}\right) \qquad \text{(move } C \text{ to the other side)}$$

$$16(y - C) = sec^{-1}\left(\frac{x^2}{8}\right) \qquad \text{(clear the fraction by multiplying)}$$

$$sec[16(y - C)] = sec\left[sec^{-1}\left(\frac{x^2}{8}\right)\right] \qquad \text{(take } sec \text{ of both sides)}$$

$$sec[16(y - C)] = \cancel{sec}\left[\cancel{sec^{-1}}\left(\frac{x^2}{8}\right)\right] \qquad \text{(cancel)}$$

$$sec[16(y - C)] = \frac{x^2}{8} \qquad \text{(simplify)}$$

$$8sec[16(y - C)] = x^2 \qquad \text{(pause here)} \tag{1}$$

Now, let's do a similar thing with the second solution, gotten by the u-substitution method:

$$y = \frac{1}{16} tan^{-1}\frac{\sqrt{x^4 - 64}}{8} + C$$

$$y - C = \frac{1}{16} tan^{-1}\frac{\sqrt{x^4 - 64}}{8} \qquad \text{(move } C \text{ to the other side)}$$

$$16(y - C) = tan^{-1}\frac{\sqrt{x^4 - 64}}{8} \qquad \text{(clear the fraction by multiplying)}$$

$$tan[16(y - C)] = tan\left[tan^{-1}\frac{\sqrt{x^4 - 64}}{8}\right] \qquad \text{(take the } tan \text{ of both sides)}$$

[1] The graphs do not include the $+C$ portion of the answers; if you read through the algebra that follows, you will see why they're not important for showing equality.

[2] There *is* a world of *inverse* trig identities, but I do not have them memorized, and each is fraught with domain restrictions, etc. that can largely be avoided by inverting those inverse functions as I will do here. (Trust me.)

$$tan[16(y-C)] = \cancel{tan}\left[\cancel{tan^{-1}}\frac{\sqrt{x^4-64}}{8}\right] \qquad \text{(cancel)}$$

$$tan[16(y-C)] = \frac{\sqrt{x^4-64}}{8} \qquad \text{(simplify)}$$

$$8tan[16(y-C)] = \sqrt{x^4-64} \qquad \text{(clear the fraction by multiplying)}$$

$$\left(8tan[16(y-C)]\right)^2 = x^4 - 64 \qquad \text{(square both sides to clear radical)}$$

$$64tan^2[16(y-C)] = x^4 - 64 \qquad \text{(square the left hand side)}$$

$$64tan^2[16(y-C)] + 64 = x^4 \qquad \text{(move the 64 to the other side)}$$

$$64\left(tan^2[16(y-C)] + 1\right) = x^4 \qquad \text{(pause here)} \qquad (2)$$

Collecting our two results from both expressions for y, we have:

$$x^2 = 8sec[16(y-C)]$$

and

$$x^4 = 64\left(tan^2[16(y-C)] + 1\right)$$

In both expressions, we see that the angle for the each of the trig functions is $16(y-C)$. Let's make a substitution of $\Theta = 16(y-C)$ to clean our expressions up a bit, leaving us with:

$$x^2 = 8sec\Theta$$

and

$$x^4 = 64(tan^2\Theta + 1)$$

Noticing that x^4 is just $(x^2)^2$, we can start the proof portion of this check, by asking:

$$(x^2)^2 \stackrel{?}{=} x^4$$

$$(8sec\Theta)^2 \stackrel{?}{=} 64(tan^2\Theta + 1) \qquad \text{(substitute)}$$

$$64sec^2\Theta \stackrel{?}{=} 64(tan^2\Theta + 1) \qquad \text{(square left side)}$$

$$64sec^2\Theta \stackrel{?}{=} 64sec^2\Theta \qquad \text{(apply Pythagorean identity)}$$

Yes! Yes, it is! Thank you, Pythagoras! For completeness, we *do* have to say that this is true for all angles Θ that do not produce $cos\Theta = 0$. That is, we have a domain restriction of, $\Theta \neq \frac{\pi}{2}(2n+1)$, for any integer, n; in words: we eliminate any odd-integer multiples of $\frac{\pi}{2}$.

Done.

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