Solutions to a couple of interesting trig equations

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Solving trigonometric equations is often an exercise in agility with various trig identities. There are also many ways to approach trig equations. Here are a couple that I did with a student recently.

Question.

Solve the following:

$$tan2x - 2cosx = 0$$

.

Solution.

For this one, I opted to convert to sines and cosines straight away. So:

$$tan2x - 2cosx = 0$$

$$\frac{sin2x}{cos2x} - 2cosx = 0 \qquad \text{(use } tan\theta = \frac{sin\theta}{cos\theta}\text{)}$$

$$\frac{sin2x}{cos2x} - \frac{2cosx \cdot cos2x}{cos2x} = 0 \qquad \text{(apply common denominator)}$$

$$\frac{sin2x - 2cosx \cdot cos2x}{cos2x} = 0 \qquad \text{(combine fractions)}$$

Now, a fraction is zero when the numerator is zero and the denominator is not zero. If we place the restriction that $cos2x \neq 0$, then we can just examine the numerator:

$$sin2x - 2cosx \cdot cos2x = 0$$

$$2sinx \cdot cosx - 2cosx \cdot cos2x = 0 \qquad \text{(expand } sin2x\text{)}$$

$$2cosx(sinx - cos2x) = 0 \qquad \text{(factor)}$$

$$cosx(sinx - cos2x) = 0 \qquad \text{(divide by 2)}$$

$$cosx(sinx + 2sin^2x - 1) = 0 \qquad \text{(expand } cos2x\text{)}$$

$$cosx(2sin^2x + sinx - 1) = 0 \qquad \text{(rearrange)}$$

$$cosx(2sinx - 1)(sinx + 1) = 0 \qquad \text{(factor the quadratic in } sinx\text{)}$$

Fully factored and equal to zero, we can take each factor, set it equal to zero, and solve:

$$cosx = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$2sinx - 1 = 0$$
$$2sinx = 1$$
$$sinx = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$sinx + 1 = 0$$
$$sinx = -1$$
$$x = \frac{3\pi}{2}$$

Remember, we placed a restriction: $cos2x \neq 0$, so we must make sure to throw out any values of x where $cos2x$ is zero:

$$cos2x \neq 0$$
$$2x \neq \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x \neq \frac{\pi}{4}, \frac{3\pi}{4}$$

Since none of the our answers were $x = \frac{\pi}{4}, \frac{3\pi}{4}$ we have five answers on $x \epsilon [0, 2\pi]$:

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

Done.

Question.

Solve the following:

$$(sin2x + cos2x)^2 = 1$$

.

Solution.

If we pause for a second, we can be a little crafty with this one and avoid a lot of terrible-looking FOILing. Straight away, we can take the square root of both sides:

$$\begin{aligned} (sin2x + cos2x)^2 &= 1 \\ \sqrt{(sin2x + cos2x)^2} &= \pm\sqrt{1} && \text{(must have the } \pm \text{ to avoid losing answers!!!)} \\ sin2x + cos2x &= \pm 1 && \text{(simplify)} \end{aligned}$$

Now, this is really two equations:

$$sin2x + cos2x = 1$$

and

$$sin2x + cos2x = -1$$

Let's solve both.

$$\begin{aligned} sin2x + cos2x &= 1 \\ sin2x + cos2x &= sin^2x + cos^2x && \text{("expand" 1)} \\ sin2x + cos^2x - sin^2x &= sin^2x + cos^2x && \text{(expand } cos2x\text{)} \\ sin2x - sin^2x &= sin^2x && \text{(cancel } cos^2x \text{ on both sides)} \\ 2sinx \cdot cosx - 2sin^2x &= 0 && \text{(expand } sin2x\text{)} \\ sinx \cdot cosx - sin^2x &= 0 && \text{(divide both sides by 2)} \\ sinx(cosx - sinx) &= 0 && \text{(factor)} \end{aligned}$$

So:

$$\begin{aligned} sinx &= 0 \\ x &= 0, \pi, 2\pi \end{aligned}$$

and:

$$\begin{aligned} cosx - sinx &= 0 \\ cosx &= sinx \\ x &= \frac{\pi}{4}, \frac{5\pi}{4} \end{aligned}$$

And now the second equation:

$$\begin{aligned}
sin2x + cos2x &= -1 \\
sin2x + cos^2x - sin^2x &= -sin^2x - cos^2x && (\text{"expand"} -1) \\
sin2x + cos^2x &= -cos^2x && (\text{cancel } -sin^2x \text{ on both sides}) \\
2sinx \cdot cosx + 2cos^2x &= 0 && (\text{expand } sin2x) \\
cosx(sinx + cosx) &= 0 && (\text{divide both sides by 2, then factor})
\end{aligned}$$

So:

$$\begin{aligned}
cosx &= 0 \\
x &= \frac{\pi}{2}, \frac{3\pi}{2}
\end{aligned}$$

and:

$$\begin{aligned}
sinx + cosx &= 0 \\
sinx &= -cosx \\
x &= \frac{3\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

Nine answers on $x\epsilon[0, 2\pi]$:

$$x = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

Done.